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small variability. Even more important than this, however, is the fact that we cannot in all strictness compare our problem with the analogous urn problem in case the test is used on more than one person, as it of course was. For the content of the dictionary, from our point of view, actually changes with the observer, depending, as it does, upon how many words *he* knows. This is an added source of variability, over and beyond that which would occur due to the ordinary errors of sampling. The result of the above paragraph is still qualitatively applicable.

Our final conclusion is, then, that the experimental evidence of the test completely justifies us in abandoning the old result and accepting the new: and that, moreover, probabilities of mixtures in the immediate neighborhood of the most probable mixture themselves follow the normal Gaussian law as given by equation (16).

A GRAPHICAL AID IN THE STUDY OF FUNCTIONS OF A COMPLEX VARIABLE.

By NORMAN MILLER, Queen's University.

The impossibility in three dimensions of representing graphically a function of a complex variable makes it necessary for the student to call on his imagination in other ways in order to realize the properties of these functions. Two methods are common in the geometrical theory of functions. One is to represent in two different planes or in two Riemann surfaces the variables z and w and to study the correspondence between the points of the two planes or surfaces, which is determined by the relation $w = f(z)$. The second method, which does much to illuminate the subject for the beginner, is to represent in one plane both the independent and dependent variables and to interpret the transformation kinematically as a flow of the points in the plane.[1]

A complete graph of the function $w = f(z)$ or $u + iv = f(x + iy)$ consists of a 2-dimensional manifold in space of four dimensions. Nevertheless the student, in his effort to visualize the function, thinks instinctively of a surface spread out over the plane of z. Such a surface is actually determined by taking for a third coördinate the absolute value of $f(z)$. Calling the third coördinate ζ the equation of the surface is

$$\zeta = \sqrt{[u(x, y)]^2 + [v(x, y)]^2},$$

only the positive square root being taken. In this representation all points on a circle of center 0 in the w-plane yield the same ordinate ζ. It is, in fact, by making no distinction among the points of such a circle that we are able to pass from a two-way spread in four dimensions to an actual surface in three dimensions.

It is interesting to enquire what properties of the function $f(z)$ are exhibited in the surface $\zeta = |f(z)|$ and what properties are lost. The following discussion will apply only to functions which are in general analytic. At a zero of such a function the surface reaches down to the plane of z and in the neighborhood of a pole it extends upward indefinitely. It will easily be seen that the following properties are among those put into evidence in the surface.

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[1] See in this connection an article by Cole, *Annals of Mathematics*, vol. 5, June, 1890.

(1) A function which is analytic over the entire plane and remains finite is a constant. (Liouville.)

If $f(z)$ is analytic in the plane without exception and the surface $\zeta = |f(z)|$ does not rise indefinitely then it can only be a plane parallel to the z-plane.

(2) The zeros of an analytic function which is not identically zero are isolated points.

The surface cannot touch the z-plane along a curve unless it is the z-plane itself. A zero may give rise to a conical point on the surface, it may belong to an edge, or it may be an ordinary point at which the z-plane is tangent.

(3) The poles of a function which is otherwise analytic are isolated points.

If from each pole a perpendicular be erected to the plane of z the surface rises indefinitely about each of these lines and nowhere else.

(4) Let $f(z)$ be any function other than a constant, analytic except for poles within and on the boundary of a region S. Then, excluding the zeros and poles, $|f(z)|$ can have neither a minimum nor a maximum at any other interior point of S. The same property may be stated more precisely as follows. In any subregion of S which includes no poles either within or on its boundary $|f(z)|$ takes on its maximum value on the boundary and in any subregion which includes no zeros $|f(z)|$ takes on its minimum value on the boundary.

If, then, a cylinder be erected on any closed curve C of S, with elements perpendicular to the z-plane it will cut from the surface a portion which will have its highest point on the boundary provided C includes no poles and which will have its lowest point on the boundary provided C includes no zeros.

(5) In the neighborhood of an isolated essentially singular point the function comes arbitrarily near to every assigned value (Weierstrass). It follows that $|f(z)|$ comes arbitrarily near to every positive real value. If a line be drawn upward from the singular point perpendicular to the plane of z then a point moving in the surface can approach as closely as we like every point on this line. The effort to visualize the surface in the neighborhood of an essentially singular point brings out clearly the nature of the singularity.

Models of the surfaces $\zeta = |f(z)|$ for some of the simpler functions would be of value in teaching the elementary theory of functions. Evidently $\zeta = |z|$ is one sheet of a circular cone with vertex 0 and axis of ζ for axis. $\zeta = |z^2|$ is a paraboloid of revolution with vertex touching the z-plane at the origin. The character of the surfaces for some other simple functions is shown in the figures 1–5. Only a portion of each surface is drawn.

(1) $\zeta = |z(z-1)|$ touches the z-plane at the points 0 and 1. Let us obtain the Cartesian equation of this surface.

$$z(z-1) = (x+iy)(x+iy-1)$$

$$= x^2 - y^2 - x + i(2xy - y),$$

$$\zeta^2 = (x^2 - y^2 - x)^2 + (2xy - y)^2.$$[1]

Translating the origin to the point $(\frac{1}{2}, 0, 0)$

$$\zeta^2 = (x^2 + y^2)^2 - \tfrac{1}{2}(x^2 - y^2) + \tfrac{1}{16}.$$

The section of the surface by a plane $\zeta = k$ is a Cassinian oval which becomes for $k = \frac{1}{4}$ the familiar lemniscate.

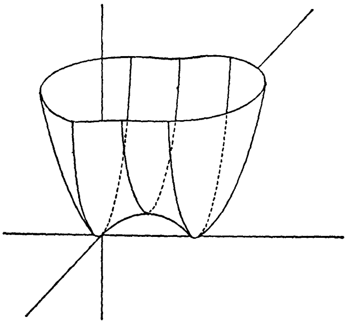

FIG. 1. $\zeta = |z(z-1)|$.

(2) $\zeta = |e^z| = e^x$. The surface is a cylinder whose elements are parallel to the axis of y and whose normal section is the exponential curve.

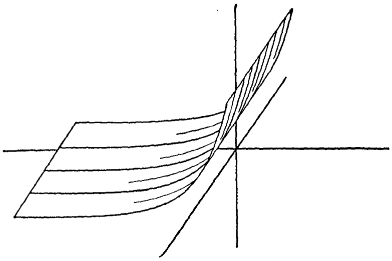

FIG. 2. $\zeta = |e^z|$.

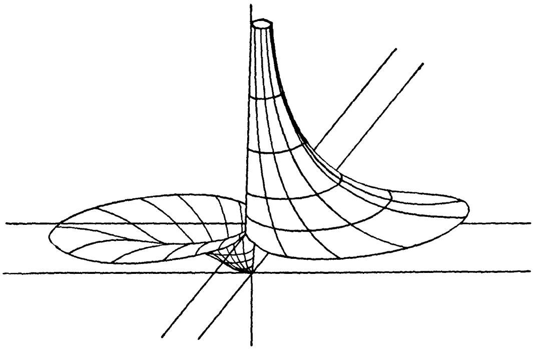

FIG. 3. $\zeta = |e^{1/z}|$.

(3) $\zeta = |e^{1/z}|$. This surface is obtained from the former one by making an inversion in the unit cylinder about the axis of ζ.[2] It consists of two parts,

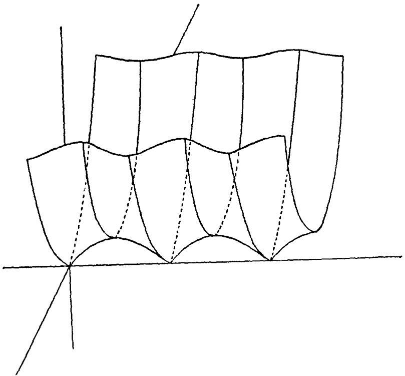

FIG. 4. $\zeta = |\sin z|$.

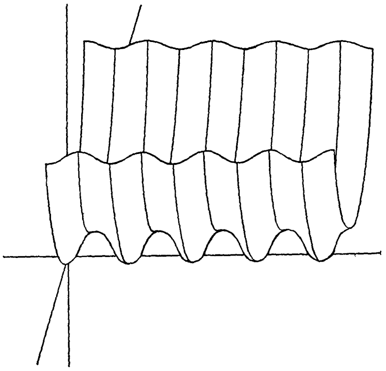

FIG. 5. $\zeta = |\sin^2 z|$.

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[1] This equation denotes 2 symmetrical surfaces of which we are concerned only with the one above the z-plane.

[2] The surface being symmetrical in the $x\zeta$-plane a reflection in this plane does not alter the character of the surface. It does of course alter the correspondence of points.

one above the plane $\zeta = 1$ and to the right of the plane $x = 0$, the other below the plane $\zeta = 1$ and to the left of $x = 0$. Each sheet of the surface approaches the plane $\zeta = 1$ asymptotically.[1] Every section of the surface by a plane parallel to the z-plane is a circle. The nature of the essential singularity at $z = 0$ is well illustrated.

(4) $\zeta = |\sin z|$ and (5) $\zeta = |\sin^2 z|$. The Cartesian equations of these surfaces are easily found to be $\zeta = \sqrt{\sin^2 x + \sinh^2 y}$ and $\zeta = \sin^2 x + \sinh^2 y$. The surfaces show the periodic character of the functions and the fact that $|\sin z|$ may take on any positive value in a period strip.

These examples are intended only to illustrate and not to exhaust the possibilities of the subject. The student should try to visualize the surface $\zeta = |\sin 1/z|$ in the neighborhood of its essential singularity. A few hours spent in constructing these surfaces will contribute towards overcoming the feeling of "a stranger in a strange land" which most students experience in entering upon the study of the theory of functions of a complex variable.

MODULAR GEOMETRY.

By ALBERT A. BENNETT, Baltimore, Md.

(Read before the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America, May 15, 1920.)

The study of relations in a single variable, in the case of finite number fields, is not, of course, a new idea, although it is awakening perhaps more interest now than in the last century. A study of several variables and in particular a characteristically geometrical treatment for finite fields may fairly be said to have been introduced within the past few years, and the only extended discussions of the whole subject to be found in any general treatise are those in Veblen and Young's *Projective Geometry,* and in a little book by G. Arnoux, *Essai de Géométrie Analytique Modulaire,* Paris, 1911. References are given in L. E. Dickson, *On Invariants and the Theory of Numbers* (Madison Colloquium Lectures), 1914, page 98.

§ 1. PLANE GEOMETRY, MODULO 3.

We shall for the sake of concreteness, consider the individual case of the field, modulo 3. This may be thought of as generated by the set of residues obtained by dividing integers by the number 3. Thus any integer, n, falls in one of three mutually exclusive classes according as n, $n - 1$, or $n - 2$ is an integral multiple of 3. We say that n is respectively congruent to 0, 1 or 2, expressed in symbols by $n \equiv 0 \ (mod\ 3)$, $n \equiv 1 \ (mod\ 3)$, $n \equiv 2 \ (mod\ 3)$, respectively. There are then only three numbers, 0, 1, 2, in the field. For these we have the following addition and multiplication tables.

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[1] Except in the $y\zeta$-plane. The plane $\zeta = 1$ intersects the surface in a line parallel to the y-axis.—EDITOR.